ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 332 (2007) 334–345
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Existence and uniqueness of solutions of semilinear stochastic infinite-dimensional differential systems with H-regular noise

H. Schurz [1]

Department of Mathematics, Southern Illinois University, 1245 Lincoln Drive, Carbondale, IL 62901-4408, USA

Received 22 November 2005

Available online 13 November 2006

Submitted by J.A. Filar

Abstract

Existence and uniqueness of approximate strong solutions of stochastic infinite-dimensional systems

$$du = \big[A(t)u + B(t,u)\big]\,dt + G(t,u)\,dW, \quad u(0,\cdot) = u_0 \in H, \quad t \geqslant 0$$

with local Lipschitz-continuous, time-depending nonrandom operators A, B and G acting on a separable Hilbert space H are studied. For this purpose, some monotonicity conditions on those operators and an existing U-series expansion of the space–time Wiener process W (U-valued, $U \subseteq H$, U Hilbert space) with $\sum_{n=1}^{+\infty} \alpha_n^2 < +\infty$ belonging to the trace of related covariance operator Q of W with local noise intensities $\alpha_n^2 \in \mathbb{R}^1$ as eigenvalues of Q are exploited.

Keywords: Stochastic partial differential equations; Strong solutions; Stochastic evolution equations; Stochastic infinite-dimensional systems; Space–time noise; Existence and uniqueness; Approximate strong solutions

1. Introduction

Consider Itô-type infinite-dimensional stochastic differential systems of the form

$$\begin{aligned} &du = \big[A(t)u + B(t,u)\big]\,dt + G(t,u)\,dW, \\ &u(0,x) = u_0(x) \in H, \quad u = u(t,x), \quad 0 \leqslant t \leqslant T,\ x \in \mathbb{D} \subset \mathbb{R}^d \end{aligned} \tag{1}$$

E-mail address: hschurz@math.siu.edu.

[1] Partially supported by University of Wyoming and Southern Illinois University.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.10.012

driven by an U-valued Wiener process W (U Hilbert space, $U \subseteq H$) in time, where A, B, G are appropriate (pseudo-)differential operators acting on the separable Hilbert space H equipped with scalar product $\langle .,. \rangle_H$. Let $\mathcal{B}(S)$ denote the σ-algebra of Borel sets of inscribed set S and μ the Lebesgue measure. We are going to prove that an approximate strong solution $u = u(t,x)$ with $\sup_{0\leqslant t\leqslant T} \mathbb{E}\|u(t,\cdot)\|_H^2 < +\infty$ for all finite, nonrandom terminal times T exists on an appropriate filtered and complete probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$. For this purpose, we suppose that the driving space–time noise W is U-regular (i.e. $\sum_{n=1}^{+\infty} \alpha_n^2 < +\infty$ where $\alpha_n^2 \in \mathbb{R}^1$ are the eigenvalues of the trace class covariance operator of W) such that

$$W = W(t,x) = \sum_{n=1}^{+\infty} \alpha_n \beta_n(t) e_n(x) \tag{2}$$

where β_n are standard independent Wiener processes and $\{e_n\colon n \in \mathbb{N}\}$ forms an orthonormal system of the Hilbert space U equipped with scalar product $\langle .,. \rangle_H$.

To be more self-explanatory, we consider the following definition of strong solution concepts. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$ be a complete probability space equipped with a nondecreasing filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T})$. Then, an H-valued stochastic process $u = (u(t))_{0\leqslant t\leqslant T}$ is said to be a *strong solution* of (1) on $([0,T] \times H \times \Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$ if

(a) u is an element of the class of progressively measurable processes with values in H (which is also closed with respect to progressively measurable versions),
(b) $u(t) \in D(A(t)) \cap D(B(t,\cdot)) \cap D(G(t,\cdot))$ ($\mathbb{P}$-almost surely) for all $t \in [0,T]$ (almost everywhere) and $A(.)u(.) \in L^1_{\mathrm{loc}}([0,T], H)$,
(c) and, for every $0 \leqslant s \leqslant t \leqslant T$, we have ($\mathbb{P}$-almost surely)

$$u(t) = u(s) + \int_s^t \big[A(r)u(r) + B(r,u)\big]\,dr + \int_s^t G(r,u)\,dW(r).$$

Moreover, an H-valued stochastic process $u = (u(t))_{0\leqslant t\leqslant T}$ is called an *approximate strong solution* of (1) on $([0,T] \times H \times \Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$ if there is a sequence of stopping times $\tau_r(t)$ with $\lim_{r\to+\infty} \tau_r(t) = t$ ($\mathbb{P}$-almost surely) such that $u_r = (u(\tau_r(t)))_{0\leqslant t\leqslant T}$ is a strong solution of (1) on $([0,\tau_r(T)] \times H \times \Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$ for all $r > 0$ and $u = \lim_{r\to+\infty} u_r \in H$ ($\mathbb{P}$-almost surely). Besides, the process $u_r = (u_r(t))_{0\leqslant t\leqslant T}$ is said to be a *localized* (strong) solution of (1). There are other solution concepts such as mild, weak and evolution solution. For more details and relations between those concepts, see Grecksch and Tudor [9]. We shall devote our studies to the concept of approximate strong solutions here.

The existence and uniqueness of strong solutions of (1) is well known when all operators are globally Lipschitz-continuous on H. In this case, a stochastic localization procedure is not needed. For example, see Da Prato and Zabzcyk [6,7], Grecksch and Tudor [9], Kotelenez [15], Rozovskii [20], Pardoux [17] or Tudor [24]. Their main results imply the existence of local pathwise unique continuous (strong) solutions $u_r \in H$ of (1) on balls

$$K_r = \big\{u \in H\colon \|u\|_H < r\big\}. \tag{3}$$

Thus, the remaining important question is how we can guarantee that u cannot explode as r tends to $+\infty$ and stays in H, i.e. our aim is to establish an existence and uniqueness result of global pathwise unique continuous (strong) solutions under conditions weaker than global Lipschitz-continuity such as local Lipschitz-continuity and further boundedness conditions. To carry such a work out, we shall borrow ideas as presented in Khasminskii [14] known for the treatment of finite-dimensional systems of ordinary stochastic differential equations (SDEs) and refer to the concept of approximate strong solutions throughout this paper.

There are several attempts to prove existence and uniqueness results for stochastic evolution equations. For example, see Bensoussan [2], Bessaih [3], Bessaih and Flandoli [4], Da Prato and Zabzcyk [6,7], Grecksch and Tudor [9], Kotelenez [15], Rozovskii [20] or Tudor [24] among many others. However, to the best of our knowledge, a general result under our main hypotheses below is not known. Perhaps, the monotonicity conditions of Pardoux [17] and Tudor [24] can be considered as the closest to ours. In fact, Pardoux [17] and [18] considers a more general class of nonautonomous, nonlinear equations

$$du(t) = \big[A\big(t, u(t)\big) + f(t)\big]\,dt + G\big(t, u(t)\big)\,dW(t) + dM(t)$$

with nonlinear operators A and G, driven by Wiener process W and martingales M. However, we shall present a different set of assumptions on these equations which allow us to exploit the interplay between linear and nonlinear parts occurring in operator A. For this purpose, we suggest to split the nonlinear operator part into a linear operator part A and a nonlinear operator part B. By this, we obtain a more efficient set of nonautonomous conditions with nonautonomous coefficients ensuring the existence and uniqueness of solutions. Moreover, we give more explicit estimates on some L^2-norm of their solutions u. Besides, our proof-technique relying on existence of local solutions and its continuation differs from that of Pardoux [17] (we believe even that our proof is simpler and more transparent, but at the expense of a fairly complex system of assumptions). Our assumptions on coercivity and monotonicity of A shall not depend on covariance operator Q of W (they are explicitly stated in terms of powers of A and without damping terms caused by G or powers of $\|u\|^p$). For example, cf. conditions of coercivity (3.6) and monotonicity (3.7) in Pardoux [18, p. 146] which depend on Q, and his coefficients α, λ and γ which do not depend on time t and our set of fully nonautonomous conditions supporting the operator splitting into A and B below (see (H2)). So, indeed we find another set of conditions on occurring operators. This is possible since we assume linearly bounded, local Lipschitz continuous diffusion operators G here in contrast to only local Lipschitz continuous G in [18]. We also allow a controlling influence of powers of the root of linear part $(-A)^{1/2}$ on the coercivity and monotonicity of operators B and G. Hence, the set of conditions of Pardoux [17,18] and ours are different, and a more detailed work out of differences we leave to the readership.

Let the domain $D(.)$ of definition of the operators A, B and D not depend on time t in order to avoid technical complications throughout this paper. Assume the validity of hypotheses that

(H0) $u_0 = u(0,x), u_0^r = u_r(0,x) \in H$ are $\mathcal{F}_0$-measurable, $\mathbb{E}[\|u_0\|_H^2 + \|u_0^r\|_H^2] < +\infty$, and $u_0, u_0^r \in D(A) \cap D(B) \cap D(G)$ ($\mathbb{P}$-almost surely).

(H1) W is a space–time Wiener process with values in a separable Hilbert space $U \subseteq H$ and covariance operator $Q : U \mapsto U$ with $\mathbb{E}[W(s) \otimes W(t)] = (t \wedge s)Q$ for $s, t \geqslant 0$, where $Q : U \mapsto U$ is a positive definite, self-adjoint, bounded operator having a finite trace $\sum_{n=1}^{+\infty} \alpha_n^2 < +\infty$ with eigenvalues α_n^2 and resulting U-converging representation

$$W = W(t)(x) = \sum_{n=1}^{+\infty} \alpha_n e_n(x) \beta_n(t) \tag{4}$$

where $(e_n)_{n\in\mathbb{N}}$ form a complete orthonormal system of H and $(\beta_n(t))_{t\geqslant 0}$ with $\beta_n(t) \in \mathcal{N}(0,t)$ are independent standard (time) real-valued Wiener processes.

(H2) The linear (unbounded) operators $A(t): D(A(t)) \subset H \mapsto H$ are self-adjoint and generate a strongly continuous two-parameter evolution semigroup $\{S(t;s):\ t \geqslant s \geqslant 0\}$ of bounded operators on H such that there are nonrandom, nonnegative coefficient functions $c_{bA}, c_{mA}, \varepsilon_{A,1}, \varepsilon_{A,2} \in L^1([0,T], \mathcal{B}([0,T]), \mu)$ satisfying

$$\langle A(t)(u), u\rangle_H \leqslant -c_{bA}(t)\|u\|_H^2 - \varepsilon_{A,1}(t)\big\|\big(-A(t)\big)^{1/2}(u)\big\|_H^2,$$
$$\langle A(t)(u-v), u-v\rangle_H \leqslant -c_{mA}(t)\|u-v\|_H^2 - \varepsilon_{A,2}(t)\big\|\big(-A(t)\big)^{1/2}(u-v)\big\|_H^2$$

for all $u, v \in D(A(t))$. (Note that one may take $c_{bA} = c_{mA}$ and $\varepsilon_{A,1} = \varepsilon_{A,2}$ due to the linearity imposed on A.)

(H3) The local Lipschitz-continuous nonlinear operators $B(t,\cdot): D(B(t,\cdot)) \subseteq H \mapsto H$ together with its Lipschitz-continuous localization $B_r(t,\cdot): D(B_r(t,\cdot)) \subseteq H \mapsto H$ are well defined and possess nonrandom coefficient functions c_0^2, c_{bB}, c_{mB}, $\varepsilon_{B,1}$, $\varepsilon_{B,2} \in L^1([0,T], \mathcal{B}([0,T]), \mu)$ satisfying

$$\langle B(t,u), u\rangle_H \leqslant c_0^2(t) + c_{bB}(t)\|u\|_H^2 + \varepsilon_{B,1}(t)\big\|\big(-A(t)\big)^{1/2}(u)\big\|_H^2,$$
$$\langle B_r(t,u), u\rangle_H \leqslant c_0^2(t) + c_{bB}(t)\|u\|_H^2 + \varepsilon_{B,1}(t)\big\|\big(-A(t)\big)^{1/2}(u)\big\|_H^2,$$
$$\langle B(t,u) - B(t,v), u-v\rangle_H \leqslant c_{mB}(t)\|u-v\|_H^2 + \varepsilon_{B,2}(t)\big\|\big(-A(t)\big)^{1/2}(u-v)\big\|_H^2,$$
$$\langle B_r(t,u) - B_r(t,v), u-v\rangle_H \leqslant c_{mB}(t)\|u-v\|_H^2 + \varepsilon_{B,2}(t)\big\|\big(-A(t)\big)^{1/2}(u-v)\big\|_H^2,$$
$$\langle B(t,u) - B_r(t,v), u-v\rangle_H \leqslant c_{mB}(t)\|u-v\|_H^2 + \varepsilon_{B,2}(t)\big\|\big(-A(t)\big)^{1/2}(u-v)\big\|_H^2$$

for all $u, v \in D(B(t,\cdot)) \cap D(B_r(t,\cdot)) \cap D(A(t))$ (usually B_r such that $D(B(t,\cdot)) \subseteq D(B_r(t,\cdot))$).

(H4) The local Lipschitz continuous, linearly bounded operators $G(t,\cdot): D(G(t,\cdot)) \subseteq H \mapsto H$ with $U \subseteq D(G(t,\cdot))$ possess nonnegative, nonrandom coefficient functions c_{bG}, c_{mG}, $\varepsilon_{G,1}, \varepsilon_{G,2} \in L^1([0,T], \mathcal{B}([0,T]), \mu)$ satisfying

$$\big\|G(t,u)\big\|_H^2 \leqslant c_{bG}(t)\big(1 + \|u\|_H^2\big) + \varepsilon_{G,1}(t)\big\|\big(-A(t)\big)^{1/2}(u)\big\|_H^2,$$
$$\big\|G(t,u) - G(t,v)\big\|_H^2 \leqslant c_{mG}(t)\|u-v\|_H^2 + \varepsilon_{G,2}(t)\big\|\big(-A(t)\big)^{1/2}(u-v)\big\|_H^2$$

for all $u, v \in D(G(t,\cdot))$.

(H5) $\forall N \geqslant N_0 \geqslant 1$ $\forall t \in [0,T]$ we have

$$2\varepsilon_{A,2}(t) - 2\varepsilon_{B,2}(t) - \varepsilon_{G,1}(t)\sum_{n=N+1}^{+\infty}\alpha_n^2 - \varepsilon_{G,2}(t)\sum_{n=1}^{N}\alpha_n^2 \geqslant 0 \quad \text{and}$$
$$2\varepsilon_{A,1}(t) - 2\varepsilon_{B,1}(t) - \varepsilon_{G,1}(t)\sum_{n=1}^{+\infty}\alpha_n^2 \geqslant 0.$$

(H6) Initial regularity of the approximation problem holds, i.e.

$$\mathbb{E}\big\|u_0 - u_0^r\big\|_H^2 \leqslant \frac{1}{2}h(r)\Big(1 + \max_{0\leqslant t\leqslant T}\mathbb{E}\big\|u(t,\cdot)\big\|_H^2\Big), \qquad \lim_{r\to+\infty} h(r) = 0, \qquad h(r) \geqslant 0.$$

In passing we note that conditions (H2) and (H3) on operators A and B are also called *coercivity and monotonicity conditions* as they arise in the theory of deterministic PDEs. Assumption (H6) also means that one can approximate the initial condition with rate function h in the mean square sense. Often one knows the initial data exactly and, in this case, only the initial regularity that $u_0 \in H$ is in the domain of definition of corresponding operators A, B and G is needed in order to talk about solvability. For standard initial value problems for ODEs or PDEs, the initial data (initial condition) are known explicitly. In this case, of course, one does not need to require the approximation rate h of the initial condition by (H6) and one may set $h \equiv 0$. However, for practical implementation of numerical algorithms, one truncates series expansions of initial data u_0 and entire solution u by u_r in finite-dimensional subspaces $H_r \subset H$. To cover this case too, we require hypothesis (H6). For all other cases, (H6) is not needed.

We start with an auxiliary result on the (possibly) stopped solution $u_r \in H$ which coincides with the solution u up to the stopping time τ_r since they exist locally.

Theorem 1. *We assume that hypotheses* (H0)–(H5) *together with $u_r(0,x) \in H$ are satisfied. Then, for any stopped* (*strong*) *solution u_r of Itô-type SDE* (1), *we have*

$$\forall 0 \leqslant t \leqslant T: \quad \mathbb{E}\big\|u_r(t,\cdot)\big\|_H^2 \leqslant \big(\mathbb{E}\big\|u_r(0,\cdot)\big\|_H^2 + K_0(T)\big)\exp\big(K_1(T)\big) \tag{5}$$

where $K_0 \geqslant 0$, $K_1 : [0,T] \to \mathbb{R}^1$ are in $L^1([0,T], \mathcal{B}([0,T]), \mu)$ and satisfy

$$K_0(T) \leqslant \int_0^T \left(2c_0^2(s) + \sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(s)\right) ds \quad \textit{and}$$

$$K_1(T) \leqslant 2\int_0^T \left(\left[-c_{bA}(s) + c_{bB}(s) + \frac{1}{2}\sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(s)\right]_+\right) ds$$

(*nondecreasing in T*) *where $[z]_+$ denotes the positive part of inscribed expression z.*

Proof. Suppose that (H0) is satisfied. For the sake of abbreviation, we drop the subscript r at u during this proof. Apply Itô formula to system (1) (see [1,10,11,13,16,19]). This implies that

$$d\|u\|_H^2 = \begin{cases} 2(\langle A(t)u, u\rangle_H + \langle B_r(t,u), u\rangle_H)\,dt + 2\langle G(t,u)\,dW_r, u\rangle_H \\ \quad + \sum_{n=1}^{+\infty} \alpha_n^2 \|G(t,u)\|_H^2\,dt. \end{cases}$$

It is not difficult to see that $\int_0^t \langle G(\cdot,u)\,dW_r, u\rangle_H$ forms a square-integrable martingale with zero expectation. Taking expectation under hypothesis (H0)–(H4) leads to the differential inequality

$$\begin{aligned} d\mathbb{E}\|u\|_H^2 \leqslant {}& -\left[2\big(c_{A,1}(t) - \varepsilon_{B,1}(t)\big) - \varepsilon_{G,1}(t)\sum_{n=1}^{+\infty}\alpha_n^2\right]\mathbb{E}\big\|\big(-A(t)\big)^{1/2}\big\|_H^2\,dt \\ & + 2\left[-c_{bA}(t) + c_{bB}(t) + \frac{1}{2}\sum_{n=1}^{+\infty}\alpha_n^2 c_{bG}(t)\right]\mathbb{E}\|u\|_H^2\,dt \\ & + \left[c_0^2(t) + \sum_{n=1}^{+\infty}\alpha_n^2 c_{bG}(t)\right]dt. \end{aligned}$$

Suppose that (H5) is satisfied. Then, the maximum solution $v(t) \geqslant \mathbb{E}\|u(t,\cdot)\|_H^2$ of above differential inequality is bounded by

$$v(t) \leqslant K_0(T) \cdot \exp\left(2\int_0^T \left(\left[-c_{bA}(s) + c_{bB}(s) + \frac{1}{2}\sum_{n=1}^{+\infty} \alpha_n^2 c_{bG}(s)\right]_+\right) ds\right)$$

where

$$K_0(T) \leqslant v(0) + 2\int_0^T c_0^2(s)\, ds + \sum_{n=1}^{+\infty} \alpha_n^2 \int_0^T c_{bG}(s)\, ds.$$

Consequently, the assertion of Theorem 1 is confirmed. □

Remark 2. The hypotheses (H2)–(H5) and the proof of Theorem 1 can be modified by using the Lyapunov functional

$$V(t,u) = \big(1 + 2c_{bA}(t)\big)\big\|u(t,\cdot)\big\|_H^2 + 2\varepsilon_{A,1}(t)\big\|\big(-A(t)\big)^{1/2}(u)\big\|_H^2,$$

and a similar assertion as in Theorem 1 is still valid, provided that initial moment $\mathbb{E}[V(0,u)] < +\infty$.

2. General theorem on existence and uniqueness

This section establishes our main result under the hypotheses (H0)–(H6) for $N \geqslant N_0$.

Theorem 3. *We assume that the hypotheses* (H0)–(H6) *are satisfied. Then, a pathwise unique continuous* (*approximate*) *strong solution u of Itô-type differential system* (1) *exists, and it is governed by the estimates*

$$\forall 0 \leqslant t \leqslant T\colon \quad \mathbb{E}\big\|u(t,\cdot)\big\|_H^2 \leqslant \big(\mathbb{E}\big\|u(0,\cdot)\big\|_H^2 + K_0(T)\big)\exp\big(K_1(T)\big), \tag{6}$$

$$\forall r \geqslant N_0\ \forall 0 \leqslant t \leqslant T\colon \quad \mathbb{E}\big\|u(t,\cdot) - u_r(t,\cdot)\big\|_H^2 \leqslant h(r)C_0(T)\exp\big(C_1(T)\big) \tag{7}$$

where $\lim_{r\to+\infty} h(r) = 0$, *and the constants $K_i(T)$ can be estimated uniformly as in Theorem* 1, *and $C_i(T)$ uniformly by*

$$C_0(T) \leqslant \Big(1 + \max_{0\leqslant t\leqslant T} \mathbb{E}\big\|u(t,\cdot)\big\|_H^2\Big) \cdot \left(1 + \int_0^T c_{bG}(t)\, dt\right),$$

$$C_1(T) \leqslant 2\int_0^T \left(\left[-c_{mA}(s) + c_{mB}(s) + \frac{1}{2}\sum_{n=1}^{+\infty} \alpha_n^2 c_{mG}(s)\right]_+\right) ds$$

(*nondecreasing in T*) *where $[z]_+$ denotes the positive part of inscribed real number z.*

Proof. Let u_r denote the stopped solution up to stopping time $\tau_r(t) = \min(\tau_r, t)$. Then it is clear from the results of Da Prato and Zabzcyk [6] (cf. Theorem 7.19, p. 213) and Grecksch and Tudor [9] (as conclusion of Theorem 2.1, p. 73) that the local (stopped) solution u_r coincides with u up to the time $\tau_r(t)$. We are going to prove that this u_r has finite H-norm as r tends to infinity, and hence the local solution can be extended to a unique global solution u. For this purpose, we borrow the technique of Lyapunov functions as indicated by Khasminskii [14] in the finite-dimensional case. Let r be sufficiently large such that $\|u(0,.)\|_H < r$. Therefore, we may set $u_r(0,x) = u_0(x)$. Furthermore, we already know that u_r must satisfy (a.s.) the truncated evolution equation

$$du_r(t) = \left[A(t)u_r + B_r(t,u_r)\right]dt + G(t,u_r)\,dW_r(t) \tag{8}$$

driven by truncated noise

$$W_r = \sum_{n=1}^{e[r]} \alpha_n \beta_n e_n$$

for $r \geqslant 1$, where $e[z]$ denotes the entire function of the largest integer which is not greater than z.

The complete proof is broken down into two major steps.

Step 1 *(Nonexplosion of unique stopped solutions).* Recall from Theorem 1 that

$$\mathbb{E}\left\|u_r(t,\cdot)\right\|_H^2 \leqslant \left(\mathbb{E}\left\|u(0,\cdot)\right\|_H^2 + K_0(T)\right)\exp\left(K_1(T)\right)$$

where $K_0 \geqslant 0$, $K_1 : [0,T] \to \mathbb{R}^1$ are in $L^1([0,T], \mathcal{B}([0,T]), \mu)$. On the other hand, by setting $V(t,u) = \|u(t,\cdot)\|_H^2$, we have

$$\begin{aligned} &r^2\mathbb{P}\left(\left\{\exists s\colon\ 0 \leqslant s < t,\ \left\|u(s,x)\right\|_H > r\right\}\right) \\ &\quad = r^2\mathbb{E}[I_{\{\tau_r < t\}}] \leqslant \mathbb{E}\left[V(t,u)I_{\{\tau_r<t\}}\right] \\ &\quad \leqslant \mathbb{E}\left[V\left(\tau_r(t),u_r\right)(I_{\{\tau_r<t\}} + I_{\{\tau_r \geqslant t\}})\right] = \mathbb{E}\left[V\left(\tau_r(t),u_r\right)\right] \end{aligned}$$

where I_S denotes the indicator function of subscribed set S. Consequently, for all $0 \leqslant t \leqslant T$, conclude that

$$\mathbb{P}\left(\left\{\exists s\colon\ 0 \leqslant s < t,\ \left\|u(s,\cdot)\right\|_H > r\right\}\right) \leqslant \frac{(\mathbb{E}\|u(0,\cdot)\|_H^2 + K_0(T))\exp(K_1(T))}{r^2}.$$

Taking the limit $r \to +\infty$ yields that

$$\mathbb{P}\left(\{\tau < T\}\right) = 0$$

where τ is the first exit time of process $\{u(t,x)\colon\ t \geqslant 0, x \in \mathbb{D}\}$ from the open set K_r. Hence, the H-norm of the local solution can never explode at finite terminal times T and the unique continuation to a unique global solution with finite H-norm must exist.

Step 2 *(Uniqueness and coincidence of limit u_∞ and u).* Let us call u_∞ the limit of the localized solutions u_r. It remains to show coincidence of u_∞ and u implying strong uniqueness too. For this purpose, subtract Eq. (8) from (1) to obtain the differential equation

$$d(u-u_r) = \left[A(t)(u-u_r) + B(t,u) - B_r(t,u_r)\right]dt \tag{9}$$

$$\qquad + G(t,u)\,d(W-W_r) + \left(G(t,u) - G(t,u_r)\right)dW_r. \tag{10}$$

Recall that $W - W_r = \sum_{n=e[r]+1}^{+\infty} \alpha_n\beta_n e_n$ and $W_r = \sum_{n=1}^{[r]} \alpha_n\beta_n e_n$ are independent and orthogonal on H. Now, apply Itô formula to (9) in order to find the differential of $\|u-u_r\|_H^2$. This yields that

$$d\|u-u_r\|_H^2 = \begin{cases} [2\langle A(t)(u-u_r), u-u_r\rangle_H + 2\langle B(t,u)-B_r(t,u_r), u-u_r\rangle_H]\,dt \\ \quad + 2\langle G(t,u)\,d(W-W_r), u-u_r\rangle_H \\ \quad + 2\langle (G(t,u)-G(t,u_r))\,dW_r, u-u_r\rangle_H \\ \quad + \|G(t,u)\|_H^2 \sum_{n=e[r]+1}^{+\infty} \alpha_n^2\,dt + \|G(t,u)-G(t,u_r)\|_H^2 \sum_{n=1}^{e[r]} \alpha_n^2\,dt. \end{cases} \tag{11}$$

Obviously, $\langle G(t,u)\,d(W-W_r), u-u_r\rangle_H + \langle (G(t,u)-G(t,u_r))\,dW_r, u-u_r\rangle_H$ forms a square-integrable martingale with vanishing expectation. Now, take the expectation at both sides of (12) and apply the monotonicity conditions (H2)–(H4) to get to the estimates

$$\begin{aligned} & d\mathbb{E}\|u-u_r\|_H^2 \\ & \quad \leqslant -\Bigg[\Bigg(2c_{A,2}(t) - 2\varepsilon_{B,2}(t) \\ & \qquad - \varepsilon_{G,1}(t) \sum_{n=e[r]+1}^{+\infty} \alpha_n^2 - \varepsilon_{G,2}(t) \sum_{n=1}^{e[r]} \alpha_n^2\Bigg) \mathbb{E}\big\|\big(-A(t)\big)^{1/2}(u-u_r)\big\|_H^2\Bigg]\,dt \\ & \qquad + 2\Bigg[\Bigg(-c_{mA}(t) + c_{mB}(t) + \frac{1}{2}\sum_{n=1}^{e[r]} \alpha_n^2 c_{mG}(t)\Bigg)\mathbb{E}\|u-u_r\|_H^2\Bigg]\,dt \\ & \qquad + \Bigg[\sum_{n=e[r]+1}^{+\infty} \alpha_n^2 c_{bG}(t)\big(1+\mathbb{E}\|u\|_H^2\big)\Bigg]\,dt. \end{aligned} \tag{12}$$

Recall hypothesis (H5) for $N = e[r] \geqslant N_0$. This leads to the differential inequality

$$dv_r(t) \leqslant \begin{cases} 2(-c_{mA}(t) + c_{mB}(t) + \frac{1}{2}\sum_{n=1}^{e[r]} \alpha_n^2 c_{mG}(t))v_r(t)\,dt \\ \quad + \sum_{n=e[r]+1}^{+\infty} \alpha_n^2 c_{bG}(t)(1+\max_{0\leqslant t\leqslant T} \mathbb{E}\|u(t,\cdot)\|_H^2)\,dt \end{cases}$$

for $v_r(t) = \mathbb{E}\|u(t,\cdot) - u_r(t,\cdot)\|_H^2$. Its maximum solution is bounded since, by Gronwall–Bellman-technique, we obtain

$$v_r(t) \leqslant \Bigg(v_r(0) + \sum_{n=e[r]+1}^{+\infty} \alpha_n^2 \Big(1 + \max_{0\leqslant s\leqslant t} \mathbb{E}\big\|u(s,\cdot)\big\|_H^2\Big) \int_0^t c_{bG}(s)\,ds\Bigg) \cdot \exp\big(C_1(t)\big)$$

where

$$C_1(t) \leqslant 2\int_0^t \Bigg(\Bigg[-c_{mA}(s) + c_{mB}(s) + \frac{1}{2}\sum_{n=1}^{e[r]} \alpha_n^2 c_{mG}(s)\Bigg]_+\Bigg)\,ds.$$

Suppose that $v_r(0) \leqslant h(r)(1+\max_{0\leqslant t\leqslant T} \mathbb{E}\|u(t,\cdot)\|_H^2)/2$ as required by (H6). Therefore, the truncation error $v(t)$ satisfies

$$v_r(t) \leqslant h(r)C_0(t)\exp\Bigg(2\int_0^t \Bigg(\Bigg[-c_{mA}(s) + c_{mB}(s) + \frac{1}{2}\sum_{n=1}^{e[r]} \alpha_n^2 c_{mG}(s)\Bigg]_+\Bigg)\,ds\Bigg)$$

where $h(r)$ satisfies (H6) and

$$C_0(t) \leqslant \Big(1 + \max_{0\leqslant s\leqslant t} \mathbb{E}\big\|u(s,\cdot)\big\|_H^2\Big)\left(1 + \int_0^t c_{bG}(s)\,ds\right).$$

Hence, as r tends to ∞, the truncation error vanishes as long as the initial data converge. Therefore, $\mathbb{P}(\{\forall t \in [0,T]\colon\ \|u(t,\cdot)\|_H \neq \|u_\infty(t,\cdot)\|_H\}) = 0$, and the assertion of Theorem 3 is proven. □

Remark 4. The hypotheses (H2)–(H5) and the proof of Theorem 3 can be modified by using the Lyapunov functionals

$$V_1(t,u) = \big(1 + 2c_{bA}(t)\big)\big\|u(t,\cdot)\big\|_H^2 + 2\varepsilon_{A,1}(t)\big\|\big(-A(t)\big)^{1/2}(u)\big\|_H^2,$$
$$V_2(t,u) = \big(1 + 2c_{mA}(t)\big)\big\|u(t,\cdot)\big\|_H^2 + 2\varepsilon_{A,2}(t)\big\|\big(-A(t)\big)^{1/2}(u)\big\|_H^2,$$

and the existence of global unique strong solutions as claimed by Theorem 3 is still guaranteed, provided that $\mathbb{E}[V_1(0,u)] < +\infty$. Moreover, as a by-product, we also obtain uniform boundedness of the form

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\big[V_1(t,u)\big] < +\infty.$$

Remark 5. Theorem 3 is important in order to carry out a reasonable analysis of numerical approximations of such class of SPDEs by eigenfunction approach. For this purpose, one truncates infinite-series solutions u of SPDEs by finite-series solutions u^N of SPDEs with finite-dimensional noise. Then the remaining problem is to find the coefficients $c_n^N(t)$ in truncation u^N by finite-dimensional approximation techniques for systems of ordinary SDEs whose L^2-error can be controlled by general theorems from [22,23]. See [21] for more details on numerical approximations of finite-dimensional systems of nonlinear SDE and [5] for related work on general rate of L^2-convergence of truncations of semilinear SPDEs.

3. An application to solutions of SPDEs with dissipative nonlinearity

For modelling nonlinear dynamics in chemical reactions and in spatio-temporal optical chaos (laserdynamics, nanotechnology) with power law (e.g. in [8] with $n = 1$ and [12] at Eq. 8.52 on p. 239), one encounters noisy reaction–diffusion equations

$$du = \big[a^2(t)\Delta u + u\big(1 - \big[\gamma(t)\big]^{2n}\|u\|_{L^2}^{2n}\big)\big]\,dt + G(t,u)\,dW$$
$$u(0) = u_0 \in L^2_{\mathrm{loc}}(\mathbb{D}) \tag{13}$$

where $n \in \mathbb{N}$, $\mathbb{D} \subseteq \mathbb{R}^d$, $\gamma : [0,+\infty) \to \mathbb{R}^1$ and $a : [0,+\infty) \to \mathbb{R}^1_+$ are certain locally Lebesgue-integrable intensity functions. Often the noise intensity $G(t,u)$ can be modelled by

$$G(t,u) = \big(\sigma_0(t) + \sigma_1(t)\|u\|_{L^2} + \sigma_2(t)\big\|(-\Delta)^{1/2}u\big\|_{L^2}\big)I_H \tag{14}$$

with real-valued noise intensities $\sigma_i(t)$ and identity operator I_H on Sobolev space $H = H^2(\mathbb{R}^d)$ with compact support. The existence and uniqueness of approximate strong solutions of (13) becomes clear from our Theorem 3. To see this, one only needs to check its assumptions (H0)–(H6). Doing so, we may take

$$A(t)u = a^2(t)\Delta u, \quad \bigl(-A(t)\bigr)^{1/2} = a(t)\nabla u,$$
$$B(t,u) = u\bigl(1 - \bigl[\gamma(t)\bigr]^{2n}\|u\|_H^{2n}\bigr), \quad B_r(t,u) = B(t,u)$$

with the domain of definition restricted to $D(A) = D(B) = H^2(\mathbb{R}^d)$. Let us check the assumptions (H2)–(H5) in some detail, whereas the assumptions (H0)–(H1) with nonrandom $u_0 = u_0^r \in H = H^2(\mathbb{D})$ are rather obvious. (H2) is satisfied since

$$\bigl\langle A(t)u, u\bigr\rangle_H = -a^2(t)\|\nabla u\|_H^2$$

under homogeneous boundary conditions on $\partial\mathbb{D}$, whereas the linearity of $A(t)$ gives the other estimate on continuity. Assumption (H3) is verified by

$$\bigl\langle B(t,u), u\bigr\rangle_H = \|u\|_H^2 - \bigl[\gamma(t)\bigr]^{2n}\|u\|_H^{2n+2} \leqslant \bigl[\gamma(t)\bigr]^{2n}\|u\|_H^2 \quad \text{and}$$
$$\bigl\langle B(t,u) - B(t,v), u - v\bigr\rangle_H = \|u-v\|_H^2 - \bigl[\gamma(t)\bigr]^{2n}\bigl\langle \|u\|_H^{2n}u - \|v\|_H^{2n}v, u - v\bigr\rangle_H$$
$$\leqslant \|u-v\|_H^2.$$

Note that $u \mapsto f(u) = (\|u\|_H^{2n})u$ is a monotonically increasing vector-valued function in u (see Lemma A.1 in Appendix A). It remains to check (H4). We have

$$\bigl\|G(t,u)\bigr\|_H^2 \leqslant 3\sigma_0^2(t) + 3\sigma_1^2(t)\|u\|_H^2 + 3\sigma_2^2(t)\bigl\|(-\Delta u)^{1/2}\bigr\|_H^2 \quad \text{and}$$
$$\bigl\|G(t,u) - G(t,v)\bigr\|_H^2 \leqslant 2\sigma_1^2(t)\|u-v\|_H^2 + 2\sigma_2^2(t)\bigl\|(-\Delta u)^{1/2} - (\Delta v)^{1/2}\bigr\|_H^2$$

for all $u, v \in D(G) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^d)$ with compact support. Now, we may set

$$\varepsilon_{A,1}(t) = a^2, \quad c_{bA}(t) = 0, \quad c_{mA}(t) = 0, \quad \varepsilon_{A,2}(t) = a^2(t),$$
$$c_0(t) = 0, \quad c_{bB}(t) = 1, \quad \varepsilon_{B,1}(t) = 0, \quad c_{mB}(t) = 1, \quad \varepsilon_{B,2}(t) = 0,$$
$$c_{bG}(t) = 3\max\bigl\{\sigma_0^2(t), \sigma_1^2(t)\bigr\}, \quad \varepsilon_{G,1}(t) = 3\sigma_2^2(t),$$
$$c_{mG}(t) = 2\sigma_1^2(t), \quad \varepsilon_{G,2}(t) = 2\sigma_2^2(t).$$

Thus, condition (H5) reads as

$$2a^2(t) - 3\sigma_2^2(t)\sum_{n=N+1}^{+\infty}\alpha_n^2 - 2\sigma_2^2(t)\sum_{n=1}^{N}\alpha_n^2 \geqslant 0 \quad \text{and} \quad 2a^2(t) - 3\sigma_2^2(t)\sum_{n=1}^{+\infty}\alpha_n^2 \geqslant 0$$

which exhibits a nontrivial interplay between diffusivity and noise intensities and reduces to the equivalent requirement

$$a^2(t) \geqslant \frac{3}{2}\sigma_2^2(t)\sum_{n=1}^{+\infty}\alpha_n^2.$$

Recall that α_n^2 are the eigenvalues belonging to the covariance operator of W. Now, additionally suppose that $u_0(x) = u(0,x) = u_r(0,x) \in H^2(\mathbb{D})$ are all nonrandom and have compact support. That also means that the initial data are all known exactly and do not need to be localized (or approximated). This is not too restrictive since it is common in most of the initial value differential problems. Hence, hypothesis (H6) is satisfied as well. Consequently, all assumptions (H0)–(H6) can be fulfilled on the Sobolev space $H = H^2(\mathbb{D})$ with compact support. Therefore, we may apply our basic result ensuring existence and uniqueness of approximate strong solutions of (13) which are nonexploding in finite time and $u \in H^2(\mathbb{D})$ with compact support. Notice that (H5) is trivially satisfied whenever $\sigma_2 \equiv 0$, otherwise when $\sigma_2^2(t) > 0$ (for all $t \geqslant 0$) sufficiently strong diffusion intensity $a^2(t)$ is needed to compensate the erratic behavior of noise terms $G(t,u)\,dW$. In the one-dimensional case $u \in \mathbb{R}^1$ with $B(t,u) = u(1 - [\gamma(t)]^{2n} u^{2n})$ instead of $u(1 - [\gamma(t)]^{2n} \|u\|_H^{2n})$ in Eq. (13), one can also verify existence of unique solutions by Theorem 3 in a similar manner. The same existence result is true for replacing $[\gamma(t)]^{2n}$ by $|\gamma(t)|^p$ with $p \geqslant 0$ in (13).

Acknowledgments

The author wishes to express his gratitude to the financial support of University of Wyoming and Southern Illinois University. It may be noted that a related problem to find the rate of convergence of truncated series approximation of the herein considered stochastic evolution equations has come up as the author visited the University of Wyoming in Laramie in November 2004 and June 2005. Special thanks go to Dr. Hakima Bessaih for numerous discussions during those visits and email correspondences, and to Dr. Werner Horsthemke and Dr. Rajarshi Roy for providing me the relevant citations of [8,12], respectively. We are grateful to the comments and suggestions of the anonymous referees leading to significant improvements.

Appendix A. Monotonicity of functions $u \in H \mapsto f(u) = \|u\|_H^p u$, $p \geqslant 0$

Lemma A.1. *Let H be a Hilbert space equipped with the real-valued scalar product $\langle .,. \rangle_H$. Then, for all $p \geqslant 0$, the mapping $u \in H \mapsto f(u) = \|u\|_H^p u$ is increasing on H and, for all $p \geqslant 0$ and all $u, v \in H$, we have*

$$g(u,v) := \big\langle f(u) - f(v), u - v \big\rangle_H \geqslant \frac{\|u\|_H^p + \|v\|_H^p}{2} \|u - v\|_H^2.$$

Proof. First, note that the above defined g is symmetric, i.e. $g(u,v) = g(v,u)$ for all $u, v \in H$. Thus, $2g(u,v) = g(u,v) + g(v,u)$. Second, we find that

$$\begin{aligned} g(u,v) &= \big\langle \|u\|_H^p u - \|u\|_H^p v + \|u\|_H^p v - \|v\|_H^p v, u - v \big\rangle_H \\ &= \|u\|_H^p \langle u - v, u - v \rangle_H + \big(\|u\|_H^p - \|v\|_H^p\big) \langle v, u - v \rangle_H \end{aligned}$$

for all $u, v \in H$. Third, both findings imply that

$$2g(u,v) = \big(\|u\|_H^p + \|v\|_H^p\big)\|u - v\|_H^2 + \big(\|u\|_H^p - \|v\|_H^p\big) \cdot \big(\|u\|_H^2 - \|v\|_H^2\big).$$

Notice that the last product term is always positive-definite. Consequently, we have

$$g(u,v) \geqslant \frac{\|u\|_H^p + \|v\|_H^p}{2} \|u - v\|_H^2$$

for all $u, v \in H$. Hence, f is increasing (in fact, $g(u,v) = 0$ or $g(u,v)$ is equal to the right side of last inequality iff $u = v$ in H). This completes the proof of Lemma A.1. □

References

[1] L. Arnold, Random Dynamical Systems, Springer-Verlag, Berlin, 1998.

[2] A. Bensoussan, Some existence results for stochastic partial differential equations, in: Stochastic Partial Differential Equations and Applications, Trento, 1990, in: Pitman Res. Notes Math. Ser., vol. 268, Longman Sci. Tech., Harlow, 1992, pp. 37–53.

[3] H. Bessaih, Martingale solutions for stochastic Euler equations, Stoch. Anal. Appl. 17 (5) (1999) 713–725.

[4] H. Bessaih, F. Flandoli, 2-D Euler equation perturbed by noise, Nonlinear Differential Equations Appl. 6 (1999) 35–54.
[5] H. Bessaih, H. Schurz, Upper bounds on the rate of convergence of truncated stochastic infinite-dimensional differential systems with H-regular noise, preprint M-04-005, Department of Mathematics, Southern Illinois University, Carbondale, 2004; Comput. Appl. Math., 2006, accepted for publication.
[6] G. Da Prato, J. Zabzcyk, Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, Cambridge, 1992.
[7] G. Da Prato, J. Zabzcyk, Ergodicity for Infinite-Dimensional Systems, London Math. Soc. Lecture Note Ser., vol. 229, Cambridge Univ. Press, Cambridge, 1996.
[8] J. García-Ojalvo, J.M. Sancho, Noise in Spatially Extended Systems, Springer-Verlag, Berlin, 1999.
[9] W. Grecksch, C. Tudor, Stochastic Evolution Equations: A Hilbert Space Approach, Akademie Verlag, Berlin, 1995.
[10] I. Gyöngy, N. Krylov, Itô formula in Banach spaces, in: Stochastic Differential Systems, Visegrád, 1980, in: Lecture Notes in Control and Inform. Sci., vol. 36, Springer-Verlag, New York, 1981, pp. 69–73.
[11] I. Gyöngy, N. Krylov, On stochastics equations with respect to semimartingales. II. Itô formula in Banach spaces, Stochastics 6 (3–4) (1981–1982) 153–173.
[12] H. Haken, Synergetics: An Introduction. Nonequilibrium Phase Transitions and Self-organization in Physics, Chemistry, and Biology, third ed., Springer-Verlag, Berlin, 1983.
[13] K. Itô, On a formula concerning stochastic differentials, Nagoya Math. J. 3 (1951) 55–65.
[14] R.Z. Khasminskii, Stochastic Stability of Differential Equations, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.
[15] P. Kotelenez, A submartingale type inequality with applications to stochastic evolution equations, Stochastics 8 (2) (1982/83) 139–151.
[16] N.V. Krylov, Introduction to the Theory of Diffusion Processes, Amer. Math. Soc., Providence, RI, 1995.
[17] E. Pardoux, Équations aux dérivées partielles stochastiques non linéaires monotones, PhD thesis, U. Paris XI, 1975.
[18] E. Pardoux, Stochastic partial differential equations and filtering of diffusion processes, Stochastics 3 (2) (1979) 127–167.
[19] P. Protter, Stochastic Integration and Differential Equations, Springer-Verlag, New York, 1990.
[20] B.L. Rozovskii, Stochastic Evolution Systems, Kluwer, Dordrecht, 1990.
[21] H. Schurz, A brief introduction to numerical analysis of (ordinary) stochastic differential equations without tears, in: D. Kannan, V. Lakshmikantham (Eds.), Handbook of Stochastic Analysis, Marcel Dekker, Basel, 2002, pp. 237–359.
[22] H. Schurz, General theorems for numerical approximation of stochastic processes on the Hilbert space H_2, Electron. Trans. Numer. Anal. 16 (2003) 50–69.
[23] H. Schurz, An axiomatic approach to numerical approximations of stochastic processes, Int. J. Numer. Anal. Model. 3 (4) (2006) 459–480.
[24] C. Tudor, On stochastic evolution equations driven by continuous semimartingales, Stochastics 23 (2) (1988) 179–195.